Dr. Pyare Lal Singh
Deptt of Mathematics
Adarsh College Rajdhanwar
Sem - VI
Paper - Core - 13 (Maths)
Mob: 9931124947

Q(a) Define complex plane

Ans. The plane whose point are represented by complex nos is called complex plane.

(b) Define Analytical fn

Ans. A function $f(z)$ is said to be Analytical Cauchy Riemann eqs are satisfied & $f'(z)$ exists.

(c) Defn continuity at a point.

Ans. A fn $f(z)$ is said to be continuous at a point if each $\epsilon > 0$, $\exists\, \delta > 0$ such that $|f(z) - f(a)| < \epsilon$ $\forall$ $|z|$

) Defn Affix of the point.

Ans. The complex nos z is known as the affix of the point (x, y), which represent it.

) Def Harmonic fn ~~is said to be~~

Ans: A fn u is said to be Harmonic if it is satisfied Laplace eqn

i.e. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(f). Q def bilinear transformation

Ans- The transformation T defined by

$w = T(z) = \frac{az+b}{cz+d}$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

where a, b, c, d are complex constants and $ad-bc \neq 0$ is called a bilinear transformat

(g) Q. Define Deferenttability at a point.

Ans: A fb $f(z)$ is said to be differentiabl if $\lim_{z \to a} \frac{f(z)-f(a)}{z-a}$ exist

Thus $f'(a) = \lim_{z \to a} \frac{f(z)-f(a)}{z-a}$ exists.

(h) Define matric space.

Ans Let X be a non-empty set. A fb of $X \times X \to R$ s.t. $d: X \times X \to R$ i called a matrix iff d satisfies the following axioms:

[m 1] : $d(x,y) \geq 0$ for all $x, y \in X$

[m 2] : $d(x,y) = 0$ iff $x = y$

[m 3] : $d(x,y) = d(y,x)$ $\forall$ $x, y \in X$

[m 4] : $d(x,y) \leq d(x,z) + d(z,y)$ $\forall$ $x, y,$

if d is a metric ~~space~~ for X, then t pair (X, d) is called a matrix spa and d is called the distance betⁿ

Q. (j) Define open sphere.

Ans. Let (X, d) be a metric space and let $a \in X$, then for any $r > 0$, the set $\{x \in X : d(x, a) < r\}$ is called an open sphere.

Q. (i) Define close sphere.

Ans. The set $\{x \in X : d(x, a) \leq r\}$ is called a close sphere centred at a and radius equal to r.

Q. (k) Def. Neighbourhood of a point in a metric space.

Ans. Let (X, d) be a metric space and $a \in X$. A subset N_a of X is called a Neighbourhood of a point $a \in X$ if there exists an open sphere $S(a, r)$ centre at a and radius r. s.t. $a \in S(a, r) \subset N_a \; \forall \; r > 0$.

Q. (l) Defn Limit point.

Ans. Let (X, d) be a metric space and let A be a subset of X. A point $x \in X$ is called a limit point if every neighbourhood of x contains a point of A other than x.

(m) Define closed set.

Ans. A set S is said to be closed if every limit point of S belongs to the set S itself.

i.e. a set is closed if $S' \subseteq S$.

Q.(n) Define subspace:
Ans: Let (X,d) be a metric space and Y be a non-empty subset of X, then the metric space (Y,d) is called a metric subspace of the metric space (X,d).

Q.(o) Define complete metric space.
Ans: A metric space (X,d) is said to be complete if every Cauchy sequence in X converge to a point of X.

Q.(p) State Banach fixed point theorem.
Ans: Let (X,d) be a complete metric space and T be a contraction mapping defined on X. Then there exists one and only one point x in X such that T(x) = x.
i.e there exists a unique fixed point in X.

Ex: Let C denote the set of all complex nrs and let a fn $d : C \times C \to R$ be defined as
$d(z_1, z_2) = |z_1 - z_2| \ \forall z_1, z_2 \in C$
Prove that d is a metric on C.

Sol: Let $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$
Then [m1]: $|z_1 - z_2| = |(x_1+iy_1) - (x_2+iy_2)|$
$= |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \geq 0$

[m2]: $d(z_1, z_2) = 0 \iff |z_1 - z_2| = 0$
$\iff |(x_1 - x_2) + i(y_1 - y_2)| = 0$
$\iff \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = 0 \Rightarrow x_1 - x_2 = 0$ & $y_1 - y_2 = 0$
$\Rightarrow x_1 = x_2$ & $y_1 = y_2 \Rightarrow (x_1 + iy_1) = (x_2 + iy_2) \Rightarrow z_1 =$

[m3]: $d(z_1, z_2) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$
$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$
$= |(x_2 - x_1) + i(y_2 - y_1)|$
$= |(x_2 + iy_2) - (x_1 + iy_1)|$
$= |z_2 - z_1| = d(z_2 - z_1)$

[m4] Let $z_1, z_2, z_3 \in C$
Then $d(z_1, z_2) = |z_1 - z_2|$
$= |(z_1 - z_3) + (z_3 - z_2)|$
$\leq |z_1 - z_3| + |z_3 - z_2|$
$= d(z_1, z_3) + d(z_3, z_2)$
$\forall z_1, z_2, z_3$

Thus all the four postulates are sati
Hence d is metric on C.

Que: Let (X, d) be any metric space, show that the f^n d^* defined by

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)} \quad \forall\, x, y \in X$$

is a metric on X.

Soln: The axioms [m1], [m2], [m3] hold because of the distance properties of the metric d.

for [m4] Let $x, y, z \in X$

$$d(x,y) \le d(x,z) + d(z,y)$$

$$\Rightarrow 1 + d(x,y) \le 1 + d(x,z) + d(z,y)$$

$$\Rightarrow \frac{1}{1+d(x,y)} \geq \frac{1}{1+d(x,z)+d(z,y)}$$

$$\Rightarrow -\frac{1}{1+d(x,y)} \le -\frac{1}{1+d(x,z)+d(z,y)}$$

$$\Rightarrow 1 - \frac{1}{1+d(x,y)} \le 1 - \frac{1}{1+d(x,z)+d(z,y)}$$

$$\Rightarrow \frac{d(x,y)}{1+d(x,y)} \le \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$$

$$\le \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$$

$$\boxed{\text{as } \frac{a+b}{1+a+b} < \frac{a}{1+a} + \frac{b}{1+b}}$$

$\therefore d^*(x,y) \le d^*(x,z) + d^*(z,y)$

Thus [m4] is satisfied.

Hence (X, d^*) is a metric space.

Bounded Metric space: definition:

Let (X,d) be a metric space, Then X is said to be bounded if there exists a positive number M s.t $d(x,y) \leq M$ for every pair of points x, y of X.

A metric space which is not bounded is said to be unbounded.

Theorem: In a metric space (X,d), finite union of bounded sets are bounded.

Proof: Let $A_1, A_2, \dots A_n$ be bounded sets in (X,d).

Then there exist $a_1, a_2, \dots a_n \in X$ and $k_1, k_2, \dots k_n > 0$ s.t

$d(x, a_m) < k_m \ \forall \ x \in A_m$; $(m = 1, 2, \dots n)$

Let $A = \bigcup_{m=1}^{n} A_m$

To prove that A is bounded.

Again let $x \in A$, Then $\in A_m$ for $(1 \leq m \leq n)$

Now, $d(x, a_1) \leq d(x, a_m) + d(a_m, a_1)$

$\leq k_m + d(a_m, a_1)$

$< \max\limits_{1 \leq m \leq n} K_m + \max\limits_{1 \leq m \leq n} d(a_m, a_1)$

$= K$ (say)

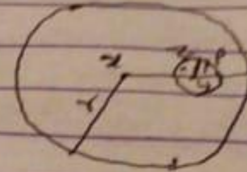
Hence A is a bounded set.

Theorem

Every open sphere in a metric space is an open set.

Proof:- Let (X, d) be a metric space.
Again let $S(x, r)$ be an open sphere in a metric space with centre $x \in X$ and radius $r > 0$. Let $y \in S(x, r)$ be arbitrary.
Let $S(y, p)$ is the open sphere with centre y and radius p.

We can prove that
$S(y, p) \subseteq S(x, r)$



We have, $d(y, x) < r$
$\Rightarrow r - d(y, x) > 0$
Let $p = r - d(y, x) > 0$

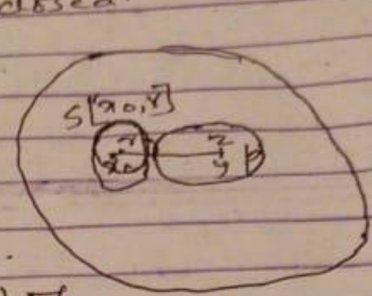
Let $z \in S(y, p)$ be an arbitrary point.
Then $d(y, z) < p$ i.e $d(z, y) < p$

Now $d(z, x) \leq d(z, y) + d(y, x)$ (by m4)
$< p + (r - p) = r$
$\Rightarrow z \in S(x, r)$
Thus $z \in S(y, p) \Rightarrow z \in S(x, r)$
Hence $S(y, p) \subseteq S(x, r)$

Thus each point of $S(x, r)$ is the centre of some open sphere contain in $S(x, r)$
Hence by defn.,
$S(x, r)$ is an open set.

Theorem. P.T. In a metric space, every closed sphere is a closed set.

Proof: Let (X, d) be a metric space and let $S[x_0, r]$ be a closed sphere in X.
We shall show that its complement
$X - S[x_0, r] = \{S(x_0, r)\}^c$ is open.
If $X - S[x_0, r] = \phi$, then we know that ϕ is open.
and hence $\{S[x_0, r]^c\}$ is closed.
Assume that $X - S[x_0, r] \neq \phi$
Let $y \in X - S[x_0, r] = \{S[x_0, r]\}^c$



Then $y \notin S[x_0; r]$ and.
So $d(y, x_0) > r$
Let p is a +ve real number.
We take an open sphere $S(y, p)$ of radius p centred at y. We now claim that the open sphere $S(y, p) \subseteq X - S[x, r]$
To verify, Let $z \in S(y, p)$ be arbitrary.
Then $d(y, z) < p$
Now, using the triangle inequality,

$$d(x_0, y) \leq d(x_0, z) + d(z, y)$$
$$< d(x_0, z) + p$$
$$\Rightarrow d(x_0, z) > d(x_0, y) - p = (r+p) - p = r$$

Thus $d(x_0, z) > r$
$\Rightarrow z \notin S[x_0, r] \Rightarrow z \in \{S[x_0, r]\}^c$
hence $z \in S(y, p) \Rightarrow z \in \{S[x_0, r]^c\}$
Which $\Rightarrow S(y, p)$ $S(y, p) \subseteq \{S[x_0, r]\}^c$,
This means the each point of $\{S[x_0, r]\}^c$ is the Some open sphere contained in $\{S[x_0, r]\}^c$
and Hence $\{S[x_0, r]\}^c = \cancel{X} - S[x_0, r]$ is open
Consequantly $S[x_0, r]$ is

Prove that every convergent sequence is bounded.

Proof. Let $\{u_n\}$ be a convergent sequence. So it must tend to finite limit. as $n \to \infty$, then by defn there exist a positive integer m depending on ϵ however small, such that

$|u_n - l| < \epsilon$, $\forall n \geq m$

i.e $l-\epsilon < u_n < l+\epsilon$ $\forall n \geq m$ —— (1)

Now take the elements $a_1, a_2, a_3 \cdots a_n$ which are finite numbers.

Let us supposed that G and g are the greatest and least element among them.

Let M = greatest of G and $l+\epsilon$
and m = least of g and $l-\epsilon$ } — (2)

from (1) and (2) we get

$m \leq u_n \leq M$, for all values of n.

Hence, by defn of bounded, the sequence $\{u_n\}$ is bounded. proved

Q. Test the convergence of the series whose general term $\sqrt{x^4+1} - \sqrt{x^4-1}$

Soln Here the nth term

$$u_n = \sqrt{x^4+1} - \sqrt{x^4-1}$$

$$= \frac{\sqrt{x^4+1} - \sqrt{x^4-1}}{\sqrt{x^4+1} + \sqrt{x^4-1}} \times \left[\sqrt{x^4+1} - \sqrt{x^4-1}\right]$$

$$= \frac{(x^4+1) - (x^4-1)}{\left[\sqrt{x^4+1} + \sqrt{x^4-1}\right]} = \frac{2}{\sqrt{x^4+1} + \sqrt{x^4-1}}$$

$$= \left[\frac{2}{\left[x^2\sqrt{1+\frac{1}{x^4}} + x^2\sqrt{1-\frac{1}{x^4}}\right]}\right]$$

$$u_n = \frac{2}{x^2\left[\sqrt{1+\frac{1}{x^4}} + \sqrt{1-\frac{1}{x^4}}\right]}$$

Let us compare the given series Σu_n which is auxiliary series Σv_n,

where $v_n = \frac{1}{x^2}$

Then $\frac{u_n}{v_n} = \frac{2}{\left[\sqrt{1+\frac{1}{x^4}} + \sqrt{1-\frac{1}{x^4}}\right]}$

$$\therefore \lim_{x\to\infty} \frac{u_n}{v_n} = \lim_{x\to\infty} \frac{2}{\left[\sqrt{1+\frac{1}{x^4}} + \sqrt{1-\frac{1}{x^4}}\right]}$$

$$= \frac{2}{1+1} = \frac{2}{2} = 1 \text{ (finite}$$

So by comparison test Σu_n & Σv_n are either convergent or both divergent.

But $\Sigma v_n = \Sigma \frac{1}{x^p}$ Here $p = 2 > 1$

Hence the given series is converg

Therefore Σu_n is also convergent.

Q. Test the convergent of series general term is $\left(1-\frac{1}{n}\right)^{n^2}$

Ans: Here $u_n = \left(1-\frac{1}{n}\right)^{n^2}$

$$\therefore (u_n)^{\frac{1}{n}} = \left\{\left(1-\frac{1}{n}\right)^{n^2}\right\}^{\frac{1}{n}} = \left(1-\frac{1}{n}\right)^{n}$$

$$= \left[\left(1-\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= (e)^{-1} = \frac{1}{e} < 1$$

Hence by cauchy's Root Test given series is convergent.